

# Effect of Cyclic Loading on the Temperature in Viscoelastic Media with Variable Properties

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Steady-state and transient temperature distributions resulting from dissipation are calculated for a linear viscoelastic slab and hollow cylinder subjected to cyclic shear loading. Temperature dependence of the dissipation function is introduced through the familiar assumption of thermorheologically simple behavior, wherein frequency-dependent mechanical properties measured at different temperatures are superposed by shifting with respect to the logarithmic frequency scale. This assumption leads to a nonlinear heat-conduction equation, and an exact closed-form solution is obtained for just the steady-state temperature distribution. In order to solve the transient problem, two approximate methods of analysis are proposed. Numerical results for a solid propellant are given, and it is found that a large temperature rise will occur as the result of a thermal instability when the shear stress is above a certain critical value that depends on thermal and mechanical properties and geometry. In this paper, inertia is neglected; however, many of the considerations, including the approximate methods, are potentially applicable to dynamic problems as well as to other configurations and loading conditions.

## Introduction

WHEN a viscoelastic solid is subjected to cyclic loading, a significant rise in temperature may occur as a result of dissipation, i.e., the conversion of mechanical energy into thermal energy. This temperature increase is roughly proportional to the square of a characteristic dimension, so that large structures are especially susceptible to dissipative heating. An extreme example of this heating was reported by Tormey and Britton<sup>1</sup> in vibration tests of solid-propellant rocket motors. As a result of vibrating a viscoelastic propellant at a resonant frequency for several hours, the temperature rise softened the propellant to such an extent that part of it flowed out of the motor.

In addition to the usual difficulties encountered in uncoupled thermoviscoelastic analysis, the characteristically strong temperature dependence of mechanical properties makes the dissipation very sensitive to temperature variations and hence leads to a strongly nonlinear equation for calculation of the temperature distribution. Although there have been some papers in the literature dealing with problems in which small temperature changes occur due to thermomechanical coupling, such as the report by Hunter,<sup>2</sup> there does not appear to be any published nonlinear analysis that includes both large temperature changes due to dissipation and the associated variations in mechanical properties.

In this paper we calculate steady-state and transient temperature distributions, in the absence of inertia, resulting from steady-state cyclic shear loading of an infinite slab with finite thickness (Fig. 1) and a long, hollow, circular cylinder (Fig. 2). Only the slab analysis is considered in detail, because it is shown that all slab results can be readily extended to the hollow cylinder by means of a simple analogy.

Starting with a general heat-conduction equation that was derived earlier from irreversible thermodynamic considerations,<sup>3</sup> we reduce it to a simple, approximate form by neglecting terms that usually have only a minor effect on the temperature. It is also assumed that the time scale for transient

temperature changes is significantly greater than a characteristic mechanical relaxation time. The resulting heat-conduction equation is first used to derive the steady-state temperature distribution in a slab. Although an exact solution to the steady-state equation is given, an exact solution in the presence of a transient temperature does not appear feasible. Two approximate methods of analysis are therefore proposed which, for comparison purposes, are first used to derive approximate steady-state solutions; reasonably good agreement with the exact result is found. We then calculate an approximate transient temperature distribution in a slab that is assumed to be initially at a uniform temperature.

Thermorheologically simple viscoelastic behavior,<sup>4</sup> which is exhibited by many linear viscoelastic solids, is assumed in the analysis. By definition, the viscoelastic mechanical properties of such a solid measured at different constant temperatures can be superposed by multiplying time or frequency by a certain temperature-dependent function, called the shift factor. The properties of a Lockheed solid propellant have been found to satisfy the assumption of thermorheologically simple behavior over a wide temperature range, and these are used in the numerical work.

For simplicity, in all of the applications, we just consider thermal boundary conditions in which one surface is insulated while the other one is maintained at a prescribed temperature. However, the analysis methods used for this case can be easily adapted to other boundary conditions. Furthermore, in some cases, the numerical results given here can be conveniently used to calculate temperatures in slabs or cylinders with other thermal boundary conditions. As a simple example of this use, it is seen that we can interpret the insulated surface of a slab as the center plane of another slab of twice the thickness and with equal surface temperatures; this latter situation corresponds to the familiar test used to obtain complex shear modulus or compliance. Likewise, additional solutions can be generated by proper adjustment of the insulated slab thickness relative to the actual slab thickness of interest, determined simply by crossplotting the present solution, with similar conclusions following for the cylinder on the basis of the analogy.

Certainly, in many engineering problems inertia cannot be neglected. However, several of the present considerations are applicable to these more involved problems, as seen

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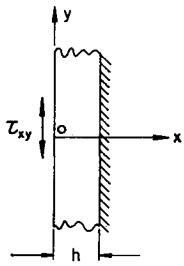


Fig 1 Viscoelastic slab

in another report<sup>5</sup> that deals with dissipative heating in the presence of inertia

### Slab Analysis

#### Governing Equation

We shall now derive an approximate equation that will be subsequently used to calculate transient and steady-state temperature fields in the linear viscoelastic slab, Fig 1. With one-dimensional heat transfer and small strains, the temperature  $T$  satisfies the partial differential equation<sup>3</sup>

$$K(\partial^2 T / \partial x^2) = Ch^2(\partial T / \partial t) - 2h^2 D \quad (1)$$

in which the nondimensional coordinate  $x = x'/h$  has been introduced.  $D$  is the dissipation function (which is proportional to the entropy production per unit volume), and the thermal conductivity  $K$  and specific heat  $C$  are assumed to be constants.† In writing this simplified form of the heat-conduction equation, we have also neglected a term that is proportional to the mechanical dilatation, since, in viscoelastic solids, this term usually has only a small influence on the temperature.‡ Consistent with these assumptions, no distinction is made between the specific heats at constant strain and at constant stress.

It is convenient, for our purposes, to express the dissipation function in terms of the local mechanical power input per unit volume and the rate of change of free energy. With simple shear loading, the dissipation is<sup>3</sup>

$$2D = \tau_{xy}(\partial \gamma_{xy} / \partial t) - (\partial V / \partial t)_T \quad (2)$$

where  $\tau_{xy}$  and  $\gamma_{xy}$  are the local shear stress and shear strain, respectively, and  $V$  is the free energy per unit volume, with the subscript  $T$  in Eq (2) denoting that the temperature is to be held constant when differentiating  $V$ . The free energy term in Eq (2) is the rate of change of "strain energy" in a viscoelastic solid, which usually produces temperature changes of only a few degrees or less, with small strains.‡ The term  $(\partial V / \partial t)_T$  will, therefore, be neglected in view of the much more significant temperature changes that can be produced by the mechanical power  $\tau_{xy} \partial \gamma_{xy} / \partial t$  under a sustained cyclic load. On the basis of these remarks, the heat-conduction equation (1) becomes

$$K(\partial^2 T / \partial x^2) = Ch^2(\partial T / \partial t) - h^2 \tau_{xy}(\partial \gamma_{xy} / \partial t) \quad (3)$$

Neglecting inertia, mechanical equilibrium considerations show that the shear stress is independent of  $x$ . Consequently it is convenient to express the shear strain  $\gamma_{xy}$  as a function

of the stress by using the linear viscoelastic stress-strain law. Consider first the stress-strain relation when the temperature is constant in time. In complex notation, a steady-state periodic shear stress of constant amplitude  $\tau_0$  and frequency  $\omega$  is given by

$$\tau_{xy} = \tau_0 e^{i\omega t} = \tau_0(\cos \omega t + i \sin \omega t) \quad (4)$$

with  $i = (-1)^{1/2}$ . Denoting the shear strain by

$$\gamma_{xy} = \gamma_0 e^{i\omega t} \quad (5)$$

the stress-strain law is

$$\gamma_{xy} = J^* \tau_{xy} \quad (6a)$$

or, equivalently,

$$\gamma_0 = J^* \tau_0 \quad (6b)$$

where  $J^*$  is the complex shear compliance

$$J^* \equiv J_1 - iJ_2 \quad (7)$$

with real part (storage compliance)  $J_1$  and imaginary part (loss compliance)  $J_2$ . Assuming that the material of interest is thermorheologically simple implies that  $J^*$  is a function of only "reduced frequency"  $\omega'$ , which is related to the actual frequency by the definition<sup>4</sup>

$$\omega' \equiv \omega a_T \quad (8)$$

with  $a_T$  being the well-known temperature-dependent shift factor that represents the effect of temperature on viscosity; the reciprocal of this factor is sometimes used.

In an exact sense, relation (6) is valid only if the temperature is not time-dependent. However, this equation will be a good approximation if  $T$ , and hence  $a_T$ , changes significantly only in time intervals that are much longer than a characteristic mechanical relaxation time. In all of the following work, it will be assumed that the temperature satisfies this condition, so that Eq (6) can be used with transient temperatures §

§ The error involved in using Eq (6) for transient temperature problems can be estimated in the following way. The exact linear viscoelastic stress-strain law in the familiar hereditary integral form is<sup>2</sup>

$$\gamma_{xy} = \int_0^t J(\xi - \xi') \frac{\partial \tau_{xy}}{\partial \tau} d\tau \quad (a)$$

where  $J(\xi)$  is the creep compliance (strain response due to a unit step-function stress), and  $\xi$  and  $\xi'$  are "reduced times" defined as

$$\xi \equiv \int_0^t \frac{dt'}{a_T} \quad \xi' \equiv \int_0^\tau \frac{dt'}{a_T} \quad (b)$$

With the periodic stress (4), and neglecting the rapidly decaying mechanical transients due to initial conditions, relation (a) can be rewritten as

$$\gamma_{xy} = J^* \tau_{xy} + i\omega \tau_{xy} \int_0^t \left[ J(\xi - \xi') - J\left(\frac{t - \tau}{a_T}\right) \right] e^{i\omega(\tau - t)} d\tau \quad (c)$$

where  $a_T = a_T[T(t)]$ . Now if a transient temperature field that is derived using the approximate equation (6) is substituted into the integral in Eq (c), and if this integral is found to be relatively small, it would be reasonable to assume that the temperature is a good approximation. Even if this error is not very small, an iteration procedure, in which the integral represents a correction term to the stress-strain equation, might be useful. When the temperature is a monotonically increasing function of time which tends to a (steady-state) constant, the approximate equation (6) may be quite accurate even for relatively high temperature rise-rates. This follows from the observation that the integral in (c) vanishes under steady-state (long-time) conditions, and both it and its first derivative are initially zero.

† In general, not only are conductivity and specific heat functions of temperature, but the latter is actually a hereditary integral operator.<sup>3</sup> The specific heat deviates from a simple coefficient mainly when the temperature is in the neighborhood of the glass-transition temperature. At such temperatures, because of the finite time required to excite the thermal degrees of freedom, the specific heat is noticeably temperature rate-dependent; characteristically for polymers, the ratio of its low-rate (or long-time) value to its high-rate (or short-time) value is about two or three.

‡ This has been verified for the solid propellant employed in the numerical work of this report.

Substitution of the real parts of  $\tau_{xy}$  and  $\partial\gamma_{xy}/\partial t$  from Eqs (4) and (6) into Eq (3) yields

$$K(\partial^2 T/\partial x^2) = Ch^2(\partial T/\partial t) - \frac{h^2\tau_0^2\omega}{2}[J_2 \cos^2\omega t - J_1 \cos\omega t \sin\omega t] \quad (9)$$

Rather than work directly with the instantaneous temperature in Eq (9), we shall integrate this expression over a cycle and then consider just the resulting equation for a cycle-averaged temperature field. We find

$$K \frac{\partial^2 \bar{T}}{\partial x^2} = Ch^2 \frac{T(t + 2\pi/\omega) - T(t)}{(2\pi/\omega)} - \frac{h^2\tau_0^2\omega J_2}{2} \quad (10)$$

with the definition

$$\bar{T} \equiv \frac{\omega}{2\pi} \int_t^{t+2\pi/\omega} T dt' \quad (11)$$

and the assumption that  $J_2$  does not change appreciably during each cycle. Use of the additional assumption that the temperature change per cycle is relatively small provides the approximation

$$\frac{T(t + 2\pi/\omega) - T(t)}{(2\pi/\omega)} \simeq \frac{\partial T}{\partial t} \quad (12)$$

If we further introduce, for convenience, the definitions of thermal relaxation time,

$$\tau_T \equiv Ch^2/K \quad (13)$$

a quantity related to dissipation,

$$\hat{D} \equiv h^2\tau_0^2\omega/2K \quad (14)$$

and omit the bar from the average temperature since it is assumed that

$$\bar{T} \simeq T \quad (15)$$

then the approximate heat-conduction equation (10) takes the desired simple form

$$\partial^2 T/\partial x^2 = \tau_T(\partial T/\partial t) - \hat{D}J_2 \quad (16)$$

We should point out that, under steady-state conditions, i.e.,  $\partial T/\partial t = 0$ , Eq (16) can be obtained without having to neglect the effect of mechanical dilatation, the operational form of specific heat, and the free energy term; these terms automatically vanish when the heat-conduction equation is integrated over a cycle.

Note that both  $\tau_T$  and  $\hat{D}$  are positive and are independent of  $x$  and  $t$ , whereas  $J_2$  is generally a very strong nonlinear, positive function of temperature through the shift factor  $a_T$ . Since Eq (16) is nonlinear, it is appropriate to examine next the behavior of  $J_2$  and  $a_T$  for a viscoelastic solid that exhibits typical properties, before solutions to Eq (16) are attempted.

### Property Characterization

We shall discuss in this section the loss compliance and shift factor of a Lockheed solid propellant, whose mechanical

and thermal properties are qualitatively typical for many viscoelastic solids.

The loss compliance in shear is shown in Fig 3 as a function of reduced frequency, using logarithmic coordinates ( $\log \equiv \log_{10}$ ). It was obtained from the tensile relaxation modulus<sup>6</sup> under the assumption of mechanical incompressibility<sup>7</sup>.

It is observed from Fig 3 that, for the intermediate reduced-frequency range  $10^{-3} < \omega a_T < 10^{12}$ ,

$$J_2 = J_n/(\omega a_T)^n \quad (\text{psi})^{-1} \quad (17)$$

with

$$n = 0.215 \quad J_n = 10^{-2.58} \simeq \frac{1}{380} \quad (18)$$

and  $\omega$  in radians per minute. Also, for lower reduced frequencies  $J_2$  is proportional to  $\omega a_T$ , whereas at higher frequencies  $J_2$  is inversely proportional to  $\omega a_T$ .

The variation of  $a_T$  with temperature, as determined from uniaxial tensile data taken at several temperatures,<sup>6</sup> is illus-

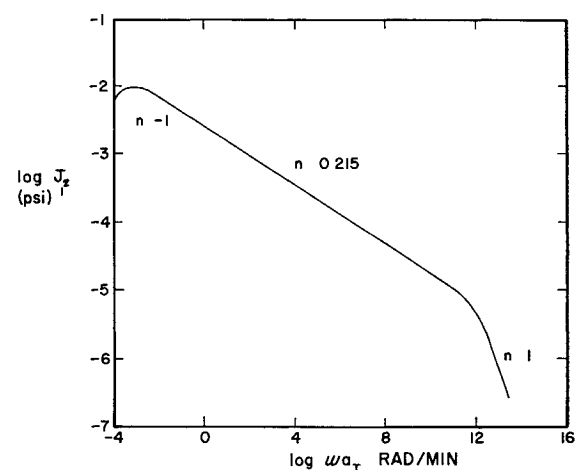


Fig 3 Loss compliance vs reduced frequency for Lockheed propellant

trated in Fig 4. We found that the experimental values could be very well approximated by the expression\*\*

$$T - T_a = T_m/a_T^m \quad (19)$$

with

$$T_a = -125^\circ\text{F} \quad T_m = 195^\circ\text{F} \quad m = 0.067 \quad (20)$$

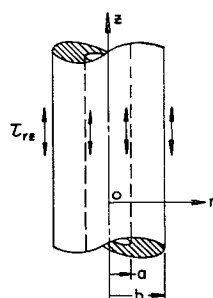
over the temperature range  $-90 < T < 120^\circ\text{F}$ . The reference temperature  $T_a$  is probably close to the glass-transition temperature; however, this latter value is not known at the present time. The analytical fit is shown by the dashed line in Fig 4.

It will be seen in the following sections that the power-law forms (17) and (19) are very convenient for analysis purposes, since they permit analytical integration in most cases. Such simplifications in the analysis would not occur if the standard Williams-Landel-Ferry (WLF) form<sup>8</sup> for  $a_T$  were used.

\* Specifically, the method of Ref 7 was used to fit a spring-dashpot model to the relaxation data (after dividing the relaxation modulus by three to convert from tension to shear), and the resulting model parameters were substituted into the complex modulus for the model. The familiar expression relating loss compliance to complex shear modulus,<sup>4</sup>  $J_2 = G_2/(G_1^2 + G_2^2)$ , where  $G_1$  and  $G_2$  are the real and imaginary parts of the complex modulus, was then used.

\*\* A power law form was also found to fit the  $a_T$  curve for several other viscoelastic materials over a wide temperature range.

Fig 2 Viscoelastic hollow cylinder



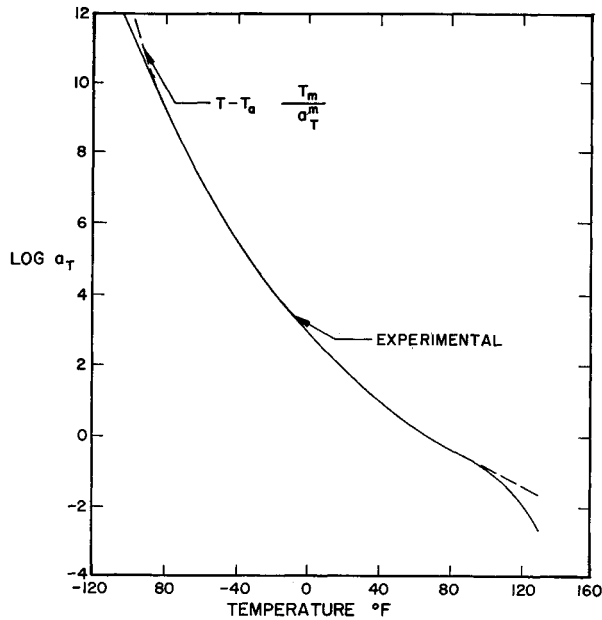


Fig 4 Shift factor vs temperature for Lockheed propellant

#### Exact Steady-State Analysis

The steady-state solution to Eq (16) satisfies the ordinary differential equation

$$d^2T/dx^2 = -DJ_2 \quad (21)$$

As boundary conditions, we assume that the temperature is prescribed at  $x = 1$ ,

$$T = T_1 \quad \text{at } x = 1 \quad (22a)$$

and that the surface  $x = 0$  is insulated,

$$dT/dx = 0 \quad \text{at } x = 0 \quad (22b)$$

Even though  $J_2$  is a function of temperature, it does not depend explicitly on  $x$ , so that Eq (21) can be solved in the closed form  $x = x(T)$ . First, multiply Eq (21) by  $dT/dx$  and integrate from 0 to  $x$ :

$$\frac{1}{2} \left( \frac{dT}{dx} \right)^2 = -\hat{D} \int_{T_0}^T J_2 dT' = \hat{D} \int_T^{T_0} J_2 dT' \quad (23)$$

where  $T_0 \equiv T(0)$  and boundary condition (22b) is used. Rearranging and integrating again, we find

$$x = \int_T^{T_0} dT' / \left( 2\hat{D} \int_T^{T_0} J_2 dT'' \right)^{1/2} \quad (24)$$

It is to be noted that  $T \leq T_0$  in view of Eq (22b) and because  $d^2T/dx^2 \leq 0$ , as seen from Eq (21).

Substitution of the special forms (17) and (19) for  $J_2$  and  $a_T$  into solution (24) yields

$$x = (\theta_1 \theta_m)^{(1/2q)-1} I_q / L \quad (25a)$$

where the following definitions are used:

$$I_q = I_q(\theta, q) \equiv \int_{\theta^{1/q}}^1 \frac{dz}{z^{1-q}} (1-z)^{1/2} \quad (25b)$$

$$\theta \equiv (T - T_a) / (T_0 - T_a) \quad (25c)$$

$$\theta_1 \equiv (T_1 - T_a) / (T_0 - T_a)$$

$$\theta_m \equiv T_m / (T_1 - T_a) \quad (25d)$$

$$q \equiv m / (m + n) \quad (25e)$$

$$L \equiv (h^2 \tau_0^2 \omega^{1-n} J_n / q K T_m)^{1/2} \quad (25f)$$

In arriving at solution (25) we have assumed, for simplicity,

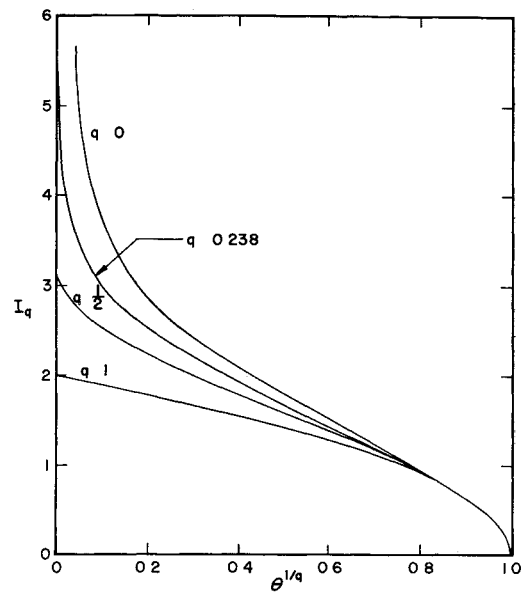


Fig 5 Temperature dependence of integral  $I_q$ , Eq (25b)

that  $n$  is independent of  $x$  and that  $q > 0$ . Of course, the results can be readily modified to account for a general piecewise constant  $n$  and  $q < 0$  ††

The integral  $I_q$  is plotted in Fig 5 for  $q = 0, \frac{1}{2}, 1$  and for  $q = 0.238$ , with the latter value corresponding to the values of  $m$  and  $n$  in Eqs (18) and (20). Observe that when  $\theta = 0$  this integral is the beta function,  $B(q, 0.5)$ :

$$I_q(0, q) = B(q, 0.5) \equiv \int_0^1 \frac{dz}{z^{1-q}} (1-z)^{1/2} \quad (26)$$

The nondimensional temperature  $\theta$  can be expressed as an explicit function of  $x$  for  $\theta^{1/q}$  close to zero and unity. When  $0 \leq \theta^{1/q} \lesssim 0.2$ , say, the square root in Eq (25b) can be omitted without introducing appreciable error, which yields the linear dependence

$$\theta \simeq q I_q(0, q) - (\theta_1 \theta_m)^{1-1/2q} q L x \quad 0 \leq \theta^{1/q} \lesssim 0.2 \quad (27)$$

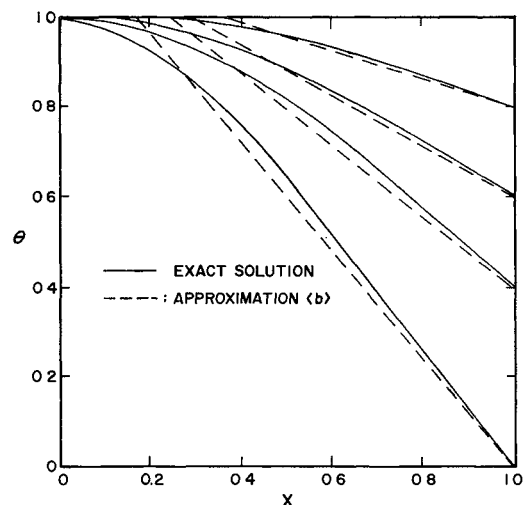


Fig 6 Nondimensional temperature distribution in slab;  $n = 0.215, m = 0.067$

†† When  $q < 0$ , definition (25b) should be replaced by

$$I_q \equiv \int_1^{\theta^{1/q}} \frac{dz}{z^{1-q}} (z-1)^{1/2}$$

and the absolute value of  $q$  should be used in Eq (25f). All other expressions in Eq (25) remain the same.

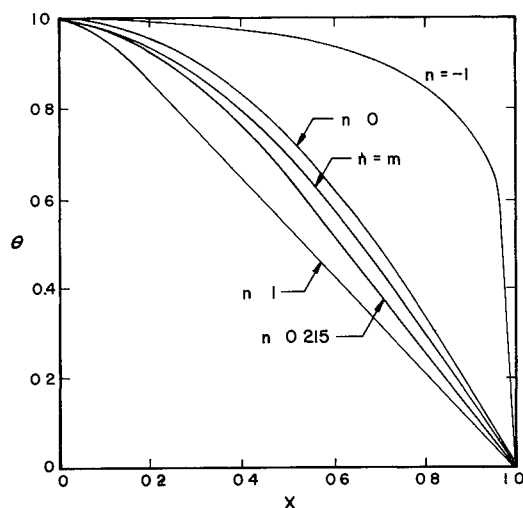


Fig 7 Comparison of nondimensional temperature distributions for several values of  $n$ ;  $m = 0.067$

On the other hand, Fig 5 shows that  $I_q$  is practically independent of  $q$  for  $0.8 \lesssim \theta^{1/q} \lesssim 1$  if  $0 \leq q \leq 1$ . Thus, setting  $q = 1$  in the integrand of Eq (25b) provides the  $x$  dependence

$$\theta \simeq [1 - (\theta_1 \theta_m)^{2-1/q} L^2 x^2 / 4]^q \quad 0.8 \lesssim \theta^{1/q} \lesssim 1 \quad (28)$$

Temperature distributions in the slab are shown in Fig 6 for  $q = 0.238$ . For this case, approximation (27) is valid for  $0 \leq \theta \lesssim 0.7$ , whereas the range of validity of Eq (28) is  $0.95 \lesssim \theta \lesssim 1.0$ .

Since the compliance curve (Fig 3) has essentially three values of  $n$  (i.e.,  $n = -1, 0.215, +1$ ), depending on the range of reduced frequency, it is also of interest to examine the effect of  $n$  on the shape of the temperature profile. This effect is most pronounced when  $\theta_1 = 0$  (i.e.,  $T_1 = T_a$ ). Thus, for comparison, distributions with this boundary value are drawn in Fig 7 with  $m = 0.067$  and several values of  $n$ . The relatively weak dependence of the shape of the curves on  $n$ , for  $0 \leq n \leq 1$ , is especially to be noted. Also, observe that, when  $n = 1$ ,  $q = 0.0628$ , so that approximation (27) is valid for practically the entire range of  $x$ . With  $n = -1$ , we find that  $0 < -q \ll 1$  and that the temperature can be approximated quite well by the expression

$$\theta \simeq [\cos\{(\theta_1 \theta_m)^{1-1/2q} Lx/2\}]^{-2q} \quad 0 < -q \ll 1 \quad (29)$$

For the case of  $n = m$ , it is found that the exact solution to

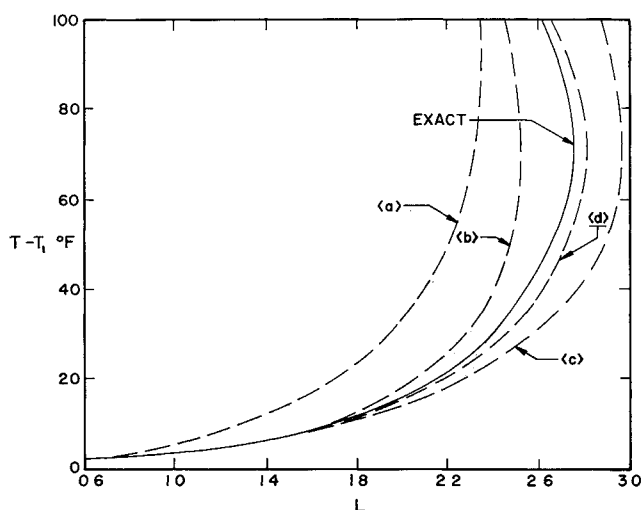


Fig 8 Temperature rise of insulated surface vs load parameter;  $T_1 = 0^\circ \text{F}$

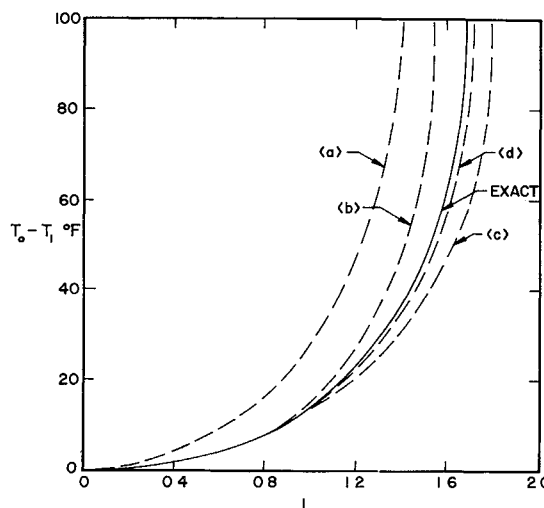


Fig 9 Temperature rise of insulated surface vs load parameter;  $T_1 = 70^\circ \text{F}$

the heat-conduction equation is

$$\theta = \cos(Lx/2) \quad m = n \quad (30)$$

Another curve shown in Fig 7 is for  $n = 0$ , namely,

$$\theta = 1 - \theta_1 \theta_m L^2 x^2 / 4 \quad n = 0 \quad (31)$$

which has the characteristic parabolic form for constant dissipation, since the loss compliance is independent of temperature when  $n = 0$ .

In order to illustrate the amount of heating which occurs, we have used solution (25) to calculate the temperature rise at the insulated surface  $x = 0$ , using the values (18) and (20), when the prescribed surface temperature is  $0^\circ$  and  $70^\circ \text{F}$ . The dependence of this temperature rise on the "load parameter"  $L$ , Eq (25f), is illustrated by the solid lines in Figs 8 and 9  $\ddagger$ .

It is especially interesting to observe that, when the load parameter is above a critical value, say,  $L_c$ , steady-state solutions do not exist. Later, we shall calculate the transient response with a supercritical value of  $L$  and show that in such a case the temperature  $T_0$  increases indefinitely. It should be recalled, however, that these results are for  $n = 0.215$  and do not take into account the reduction in loss compliance which occurs when the temperature becomes high enough (small reduced frequency) to require the use of  $n = -1$ . In fact, when the reduced frequency is small enough to use  $n = -1$  over the entire slab, the applicable equation (29) can be used to show that a steady-state solution will exist if  $\theta_1 > 0$ . Nevertheless, in view of the broad frequency and temperature range for which the value  $n = 0.215$  is valid, a large amount of heating can occur when  $L \gtrsim L_c$ .

The dependence on surface temperature  $T_1$  of the critical load parameter  $L$  and corresponding critical temperature rise  $(T_0 - T_1)$  is obtained from the condition

$$\partial L / \partial T_0 = 0 \quad (32)$$

$\ddagger$  Knowing  $T_0$ , and hence  $\theta_1$ , we can calculate the relative displacement  $\Delta v$  between the faces  $x = 0$  and  $x = 1$ . This displacement is conveniently expressed in terms of an average complex compliance  $\bar{J}^* \equiv \Delta v / \tau_{xy} h$ . With  $J_1/J_2$  constant,

$$\bar{J}^* = [J^*]_{T=T_1} \{ 2\theta_m^{(1/2q)-1} (\theta_1^{-1/q} - 1)^{1/2} / L \} = [J^*]_{T=T_1} \{ 2\theta_1^{1-1/q} (1 - \theta_1^{1/q})^{1/2} / I_q(\theta_1, q) \}$$

Note that this expression also provides a correction factor that is useful when the slab is employed for the experimental determination of  $J^*$ .

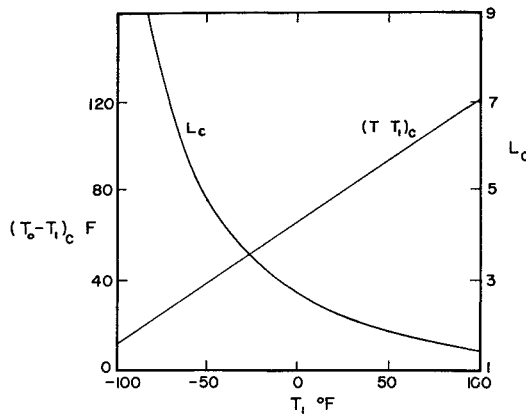


Fig. 10 Critical temperature rise and load parameter vs surface temperature

Use of Eq (32), in conjunction with Eq (25a) evaluated at  $x = 1$ , yields an expression for the critical value of  $\theta_1$ :

$$I_q(\theta_1, q) = \frac{2\theta_{1c}}{(1 - 2q)(1 - \theta_{1c}^{1/q})^{1/2}} \quad (33)$$

where  $\theta_{1c}$  is the value of  $\theta_1$  when  $L = L_c$  and is seen to be a function of only the constant  $q$ . Note that with specified values of material properties the critical temperature rise  $(T_0 - T_1)_c$  is linear in the surface temperature  $T_1$  according to definition (25c); thus,

$$(T_0 - T_1)_c = (T_1 - T_a)[(1/\theta_{1c}) - 1] \quad (34)$$

The critical value  $L$  is now found by substituting Eq (33) into Eq (25a) and setting  $x = 1$  and  $\theta_1 = \theta_{1c}$ :

$$L_c = \frac{2(\theta_m \theta_{1c})^{1/2q}}{\theta_m(1 - 2q)(1 - \theta_{1c}^{1/q})^{1/2}} \quad (35)$$

which can be expressed in terms of surface temperature  $T_1$  and material properties by using definition (25d); thus,

$$L = \frac{2\theta_{1c}^{1/2q}}{(1 - 2q)(1 - \theta_{1c}^{1/q})^{1/2}} \left( \frac{T_1 - T_a}{T_m} \right)^{1-1/2q} \quad (36)$$

It is found from Eq (33) that critical values of the load parameter exist only for the range  $0 < q < \frac{1}{2}$ , and that  $\theta_{1c} \rightarrow 1$  as  $q \rightarrow 0$  and  $\theta_{1c} \rightarrow 0$  as  $q \rightarrow \frac{1}{2}$ . The parameter  $q$  represents, therefore, a measure of the stability of a material, with the smaller values in the forementioned range corresponding to a less stable one. Furthermore, all steady-state solutions for which  $0 < q < \frac{1}{2}$  are double-valued over the entire range  $0 < L < L_c$ , and  $T_0 \rightarrow \infty$  as  $L \rightarrow 0$  on the upper branch of  $T_0(L) - T_1$ . However, the upper branch does not appear to be physically acceptable, since an increase in load parameter causes a decrease in steady-state temperature §§

Critical values of temperature rise (34) and load parameter (36) are plotted in Fig. 10 as a function of surface temperature  $T_1$  for the properties in Eqs (18) and (20). For these properties  $q = 0.238$ , and we find  $\theta_{1c} = 0.65$ ; thus, only those profiles in Fig. 6 for which  $\theta_1 > 0.65$  represent physically admissible steady-state solutions.

Finally, it is of interest to point out that the behavior of the steady-state temperature is similar to that found by Copple et al.<sup>10</sup> in their investigation of the heating of solid dielectrics and by Gruntfest<sup>11</sup> in his study of plane viscous flow.

§§ It can be easily shown, using a standard perturbation technique, that the upper branch is unstable in the neighborhood of the critical point; analytical proof of instability over the entire range appears to be very involved.

### Approximate Steady-State Analysis

We have seen that, because of the particularly simple form of the steady-state equation (21), it could be solved analytically as  $x = x(T)$ . On the other hand, such a simple solution does not exist when the temperature is time-dependent or when inertia is included.<sup>5</sup> It is appropriate, therefore, to consider the use of approximate methods of analysis which can be conveniently employed when exact solutions are not available.

In this section, we briefly examine the applicability of an integral method and a variational principle for the quasi-static, steady-state problem. Since the exact solution for this case has just been obtained, it will serve as a basis for comparing the accuracy of the two approximate methods relative to their ease of application. Extensions of these methods to the transient problem will then be made in the next section. We should add that there have been several approximate techniques proposed for solving various types of heat-conduction problems, which can be extended to the present nonlinear dissipation problem, and an extensive bibliography is given in Ref. 12. However, there will be no attempt here to evaluate the relative merits of each one, but we shall just outline the two forementioned methods, which appear to be particularly suited to the dissipation problem, and then present a numerical application of each one.

The procedure for both the integral and variational techniques will be to assume a temperature distribution that satisfies boundary conditions and consists of a prescribed function of  $x$  with some arbitrary parameters. These parameters are calculated such that the heat-conduction equation (21) is satisfied in some weighted-average sense, with the weighting function depending on the method employed.

Consider first the heat-balance integral technique proposed by Goodman.<sup>13</sup> If there is only one free parameter in the approximate temperature distribution, it is calculated by satisfying Eq (21) integrated over the slab:

$$(dT/dx)_1 = -\hat{D} \int_0^1 J_2 dx \quad (37)$$

where boundary condition (22b) has been used. This integral condition represents physically a balance between the heat generated from dissipation and the heat leaving the slab. When two parameters are included in the assumed temperature, another condition is required, and a natural, easily applied extension of Eq (37) is to require that the second integral of Eq (21) be satisfied; thus,

$$T_1 - T_0 = -\hat{D} \int_0^1 \int_0^x J_2 dx' dx \quad (38a)$$

which, when integrated by parts and combined with condition (37), can be shown to be equivalent to the first moment of Eq (21):

$$\int_0^1 \left( \frac{d^2 T}{dx^2} + \hat{D} J_2 \right) x dx = 0 \quad (38b)$$

Even though this procedure can be generalized by taking higher moments of the heat-conduction equation when additional free parameters are used, it rapidly becomes very cumbersome because of the nonlinear form of  $J_2$ .

Consider now a variational principle that will be used in conjunction with the well-known Ritz method<sup>14</sup> to obtain an approximate solution. First, we define the functional  $F$ ,

$$F \equiv \int_0^1 \left[ \frac{1}{2} \left( \frac{dT}{dx} \right)^2 - \hat{D} \int^T J_2 dT' \right] dx \quad (39)$$

and then state the variational principle: among all temperature distributions that satisfy boundary conditions (22), the actual temperature is determined by making  $F$  stationary with respect to all possible small variations  $\delta T$  of the tem-

perature from its actual value. That is, the Euler equation of

$$\delta F = 0 \quad (40)$$

is Eq (21). To verify this principle, we carry out the variation (40) and find

$$-\delta F = \int_0^1 \left[ \frac{d^2 T}{dx^2} + \hat{D} J_2 \right] \delta T dx = 0 \quad (41)$$

which requires that the integrand vanish since  $\delta T$  is arbitrary.

It is interesting to observe that this principle is analogous to Hamilton's principle in dynamics for the motion of a body,<sup>15</sup> with the first term in the functional  $F$  corresponding to the kinetic energy and the second term appearing as the potential energy. The functional  $F$  also has direct thermodynamic significance in that it is proportional to the difference between entropy production arising from temperature gradients and an integral of that due to mechanical dissipation.

Applying now the integral technique to the problem that was solved in the previous section, we assume a temperature distribution that is sufficiently simple to permit analytical integration of conditions (37) and (38). One such distribution is the linear form

$$\theta = 1 \quad \text{for} \quad 0 \leq x \leq x_1 \quad (42a)$$

$$\theta = \alpha - x(\alpha - \theta_1) \quad \text{for} \quad x_1 \leq x \leq 1 \quad (42b)$$

in which continuity at the point  $x_1$  requires

$$x_1 = (\alpha - 1)/(\alpha - \theta_1) \quad (43)$$

$$L = \frac{2^{1/2}(1+q)^{1/2}(\theta_1\theta_m)^{(1/2q)-1}[1 - \theta_1(1+\alpha)/2\alpha]^{1/2}(\alpha - \theta_1)}{q\{([1 - \alpha]/\alpha q)(\theta_1 - \alpha)(1+q) + 1 - \theta_1 - q\theta_1(1 - \theta_1^{1/q})\}^{1/2}} \quad (48)$$

and definition (25c) relates the dimensionless temperatures  $\theta$  and  $\theta_1$  to the actual temperatures. With the temperature  $T_1$  given, there are two presently unprescribed parameters in Eq (42), and we shall take  $T_0$  and  $\alpha$  as these parameters.

Substitution of the temperature (42) into the heat-balance integral (37) yields one condition:

$$L = \frac{2^{1/2}(\theta_1\theta_m)^{(1/2q)-1}(\alpha - \theta_1)}{q[1 - \theta_1^{1/q} + (\alpha - 1)/q]^{1/2}} \quad (44)$$

If  $\alpha$  is given, this condition can be used to calculate the insulated surface temperature  $T_0$ . For comparison purposes, the approximate temperature  $T_0$  has been calculated as a function of  $L$ , with  $\alpha = 1$ , for  $T_1 = 0^\circ$  and  $70^\circ\text{F}$ , using the

Lockheed propellant properties, and the results are indicated in Figs 8 and 9 by the symbol  $\langle a \rangle$ . Observe that  $x_1$  vanishes when  $\alpha = 1$ , according to Eq (43), and hence the rather poor results for this case are not surprising.

A second, and better, approximation can be obtained by using condition (38a) in order to evaluate  $\alpha$ . After substituting the temperature (42) into Eq (38a) and making use of condition (44), we find the second condition

$$(\alpha - 1)^2 = \frac{q^2}{1+q} - q\theta_1^{1/q} \left( 1 - \frac{\theta_1}{1+q} \right) \quad (45)$$

The insulated surface temperatures calculated from Eqs (44) and (45) are illustrated in Figs 8 and 9 and are designated by  $\langle b \rangle$ . A few of the associated profiles are shown in Fig 6.

The variational principle can also be conveniently used to evaluate  $\alpha$  and  $T_0$ , and the stationary requirement (40) provides the two conditions

$$\frac{\delta F}{\delta T_0} = \frac{1}{2} \frac{\partial}{\partial T_0} \left\{ \int_0^{x_1} \left( \frac{dT}{dx} \right)^2 dx + \int_{x_1}^1 \left( \frac{dT}{dx} \right)^2 dx \right\} - \hat{D} \left\{ \int_0^{x_1} J_2 \frac{\partial T}{\partial T_0} dx + \int_{x_1}^1 J_2 \frac{\partial T}{\partial T_0} dx \right\} = 0 \quad (46)$$

$$\frac{\delta F}{\delta \alpha} = \frac{1}{2} \frac{\partial}{\partial \alpha} \left\{ \int_0^{x_1} \left( \frac{dT}{dx} \right)^2 dx + \int_{x_1}^1 \left( \frac{dT}{dx} \right)^2 dx \right\} - \hat{D} \left\{ \int_0^{x_1} J_2 \frac{\partial T}{\partial \alpha} dx + \int_{x_1}^1 J_2 \frac{\partial T}{\partial \alpha} dx \right\} = 0 \quad (47)$$

Condition (46) yields

which is considerably more involved than the corresponding expression (44) obtained by the heat-balance method. When  $\alpha = 1$ , solution (48) becomes

$$L = \frac{2^{1/2}(\theta_1\theta_m)^{(1/2q)-1}(1 - \theta_1)}{q[1 - q(1 - \theta_1^{1+1/q})/(1+q)(1 - \theta_1)]^{1/2}} \quad (49)$$

and is seen to be quite similar to Eq (44) for  $\alpha = 1$ . The insulated surface temperatures predicted by Eq (49) are shown in Figs 8 and 9 and are denoted by  $\langle c \rangle$ .

From the second condition (47), there results

$$L = \frac{(1+q)^{1/2}(\theta_1\theta_m)^{(1/2q)-1}(\alpha - \theta_1)}{q[1 - q\theta_1(1 - \theta_1^{1/q})/(1 - \theta_1)]^{1/2}} \quad (50)$$

which, when combined with Eq (48), determines  $T_0$  and  $\alpha$ . Resulting surface temperatures are indicated by  $\langle d \rangle$  in Figs 8 and 9. Also, for comparison, we have plotted in Fig 11 the values of  $x_1$  for both solutions  $\langle b \rangle$  and  $\langle d \rangle$ .

It is seen from Figs 8 and 9 that the two-parameter variational solution  $\langle d \rangle$  is significantly better than the two-parameter solution  $\langle b \rangle$ , and that the quality of the latter is about the same as  $\langle c \rangle$ . The two-parameter variational result is, however, obtained at the expense of more algebra. Furthermore, all four solutions exhibit the same qualitative behavior as the exact solution.

Considering the fact that a rather simple form was assumed for the temperature distribution, the results obtained in this section are very favorable. On this basis we shall now extend these methods to the transient problem, whose exact solution is not available.

#### Approximate Transient Analysis

Both approximate methods are easily adapted to the calculation of transient temperatures that are governed by the differential equation (16). For the heat-balance integral

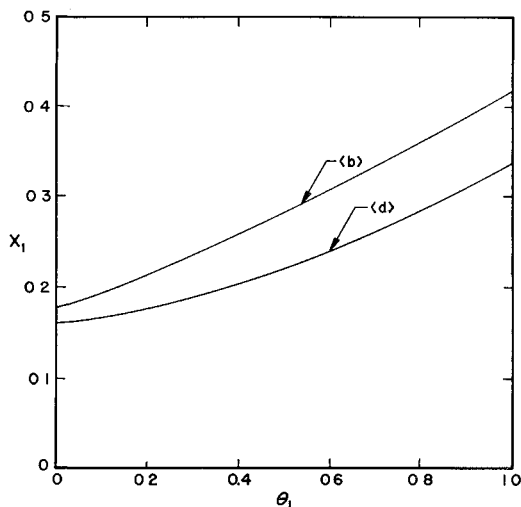


Fig 11 Dependence of  $x_1$  on nondimensional surface temperature

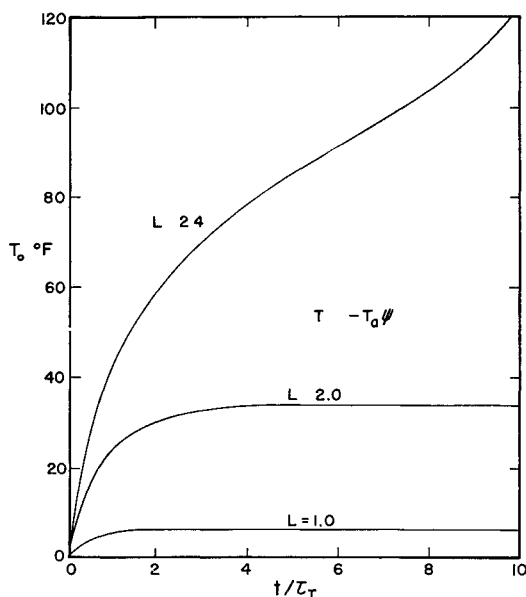


Fig. 12 Time dependence of insulated surface temperature from heat-balance integral method;  $\alpha = 1$ ,  $T_1 = 0^\circ\text{F}$

technique, we integrate Eq (16) over the slab to obtain

$$\left(\frac{\partial T}{\partial x}\right)_1 = \tau_r \frac{d}{dt} \int_0^1 T dx - \hat{D} \int_0^1 J_2 dx \quad (51)$$

which replaces the steady-state condition (37). Condition (51) equates the heat leaving the slab to the difference between dissipation and the rate of change of internal energy. Similarly, generalization of the second steady-state condition is made by adding the double integral of  $\tau_r \partial T / \partial t$  to the right-hand side of Eq (38a).

The variational principle is extended to the transient problem by simply modifying the condition for stationary behavior (40) so that it becomes

$$\delta F + \tau_r \int_0^1 \frac{\partial T}{\partial t} \delta T dx = 0 \quad (52)$$

with

$$F \equiv \int_0^1 \left[ \frac{1}{2} \left( \frac{\partial T}{\partial x} \right)^2 - \hat{D} \int^T J_2 dT' \right] dx \quad (53)$$

The integral and variational methods can be used to calculate transient temperatures in a manner analogous to the procedure used for the steady-state problem. However, there is an additional required step of solving differential equations in time. Let us give a simple illustration of a transient analysis by substituting the approximate temperature (42) with  $\alpha = 1$  into the heat-balance integral (51) and calculate time dependence of the insulated surface temperature  $T_0$ . It will be assumed that the slab is initially at the temperature  $T_1$  and that the surface  $x = 1$  is maintained at this temperature. This numerical example will show the nature of the instability that exists when a supercritical value of  $L$  is used.

Making the forementioned substitution, we obtain the following differential equation:

$$\tau_r (d\psi/dt) + 2\psi - (L^2 q^2 \theta_m^{2-1/q} / \psi) \times [(\psi + 1)^{1/q} - 1] = 0 \quad (54)$$

with the definition

$$\psi \equiv (T_0 - T_1) / (T_1 - T_a) \quad (55)$$

and the initial condition  $\psi(0) = 0$ . The solution to Eq (54)

is

$$\frac{t}{\tau_r} = \int_0^\psi \left\{ \frac{L^2 q^2 \theta_m^{2-1/q}}{\psi'} [(\psi' + 1)^{1/q} - 1] - 2\psi' \right\}^{-1} d\psi' \quad (56)$$

which is plotted in Fig. 12 for  $T_1 = 0^\circ\text{F}$  and the properties in Eqs (18) and (20). Since the corresponding critical load parameter is  $L \approx 2.36$ , as seen from curve (a) in Fig. 8, the divergent response in Fig. 12 results from a supercritical load parameter. Although solution (56) is not expected to be quantitatively accurate, in view of the poor steady-state results for  $\alpha = 1$ , the qualitative behavior is in agreement with temperatures given in Refs. 10 and 11; in both of these investigations, analog computers were used to solve a transient heat-conduction equation that is similar to the present one.

When two independent parameters are used, both the integral and variational methods lead to differential equations that cannot be solved in the simple form (56). However, by making a minor approximation, it is possible to express the solution as a single integral and obtain results that are in good agreement with analog-computer solutions.<sup>5</sup>

## Cylinder Analysis

### Governing Equations

It will now be shown that all of the slab results can be extended to the hollow cylinder in Fig. 2 by means of a simple analogy.

The heat-conduction equation for cylindrical coordinates, corresponding to the slab equation (16), is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = \frac{Cb^2}{K} \frac{\partial T}{\partial t} - \frac{b^2 \omega J_2}{2K} \frac{\tau_1^2}{r^2} \quad (57)$$

where the dimensionless radius  $r \equiv r'/b$  is used and where  $\tau_1$  is the shear stress at the outer boundary,  $r = 1$ . The radial dependence of the dissipation term in Eq (57) is because the local shear stress  $\tau_r$  is given by

$$\tau_r = \tau_1 / r \quad (58)$$

in order to satisfy mechanical equilibrium.<sup>16</sup> Also, the contribution to dissipation of all components of the stress tensor other than  $\tau_r$  is neglected in Eq (57); these other stresses and corresponding strains arise from thermal expansion and may lead to a small amount of dissipation in the presence of a transient temperature.

As boundary conditions, it is assumed that the inner boundary is insulated and that the temperature at the outer boundary is given, viz.,

$$T = T_1 \quad \text{at} \quad r = 1 \quad (59a)$$

$$\partial T / \partial r = 0 \quad \text{at} \quad r = a/b \quad (59b)$$

We now change the radius variable to the new coordinate  $y$ , using the transformation

$$r = e^{y \ln \lambda} / \lambda \quad (60)$$

with the inverse

$$y = \ln r \lambda / \ln \lambda \quad (61)$$

where  $\lambda \equiv b/a$ . Observe that at the inner boundary  $y = 0$ , and at the outer boundary  $y = 1$ . Applying this transformation to Eq (57), we find

$$\partial^2 T / \partial y^2 = \tau_r' (\partial T / \partial t) - \hat{D}' J_2 \quad (62)$$

with the definitions

$$\tau_r' \equiv (a \ln \lambda)^2 (C/K) e^{2y \ln \lambda} \quad (63)$$

$$\hat{D}' \equiv (\tau_1^2 \omega / 2K) (b \ln \lambda)^2 \quad (64)$$



The boundary conditions (59) transform to

$$T = T_1 \quad \text{at} \quad y = 1 \quad (65a)$$

$$\partial T / \partial y = 0 \quad \text{at} \quad y = 0 \quad (65b)$$

### Analogy

Upon comparing Eq (62) and boundary conditions (65) with the corresponding slab equations (21) and (22), it is clear that all of the steady-state slab results can be immediately extended to the hollow cylinder if we make the changes

$$x \rightarrow \ln r \lambda / \ln \lambda \quad h \rightarrow b \ln \lambda \quad \tau_0 \rightarrow \tau_1 \quad (66)$$

For the transient problem the analogy is not complete because  $\tau_T'$  in Eq (63) is a function of the coordinate, but the slab value  $\tau_T$  [Eq (13)] is constant. However, when one of the approximate methods discussed in the previous section is employed, a direct correspondence can often be made; it can be easily shown that such an analogy results by taking either  $\alpha$  or  $x_1$  in Eqs (42) and (43) as a constant. For example, applying the heat-balance integral method to Eq (62), with the assumed temperature given by Eq (42) with  $\alpha = 1$ , we find that the transient solution given in the previous section is applicable if, in addition to the changes (66), we make the substitution

$$\tau_T \rightarrow (Ca^2/2K)(\lambda^2 - 1 - 2 \ln \lambda) \quad (67)$$

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